Chapter 1

Partial Fractions

In this unit, students will:
(a) learn how to recognize a proper and improper fraction;
(b) know that a proper fraction can be rewritten as a sum of partial fractions;
(c) know how to use the three rules of partial fractions.

1.1 Introduction

A polynomial *p(x)* in *x*, is an expression of the form $p(x) = a_o + a_1x + a_2x^2 + ... + a_nx^n$ where $n \in Z^+ \cup \{0\}$. The degree of the polynomial is *n*.

An **algebraic fraction** is an expression of the form $\frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials. It can often be rewritten as the **sum of simpler fractions** that are called **partial fractions**. For example, it can be shown that $\frac{3x+1}{2x^2-x-1}$ can be written in partial fractions as $\frac{1}{3(2x+1)} + \frac{4}{3(x-1)}$

Proper and Improper Fractions

When the degree of the numerator, that is the highest power in *x* for the polynomial on top of the fraction, is strictly less than the degree of the denominator, that is the highest power in *x* for the polynomial on the bottom of the fraction, the fraction is said to be **proper**.
The fraction $\frac{x-1}{x^2-2x+1}$ satisfies this condition and so is proper.

If a fraction is not proper it is said to be **improper**. For example, the fraction $\frac{x^3+1}{x^2-3x+2}$ is improper because the degree of the numerator, 3, is greater than the degree of the denominator, 2.

The first stage in the process of finding partial fractions is to determine whether the fraction is proper or improper because proper fractions are simpler to deal with.

1.2 Finding partial fractions of improper fraction

$\frac{2x^3+3x^2-x+1}{x^2-x-2}$ is an improper fraction, we convert it to a sum of polynomial and a proper fraction by long division:

$$\begin{array}{r} 2x+5 \\ x^2-x-2\overline{\big)\,2x^3+3x^2-x+1} \\ -(2x^3-2x^2-4x) \\ \hline 5x^2+3x+1 \\ -(5x^2-5x-10) \\ \hline 8x+11 \end{array}$$

$$\therefore \quad \frac{2x^3+3x^2-x+1}{x^2-x-2} \equiv (2x+5)+\frac{8x+11}{x^2-x-2}$$

where $\frac{8x+11}{x^2-x-2}$ is a proper fraction which can be resolved into partial fractions which is explained in section 1.3

Exercise 1.1

(i) Express $\frac{x^3-1}{(x+1)(x+2)}$ in proper fraction. $\left[\frac{x^3-1}{(x+1)(x+2)} = (x-3)+\frac{7x+5}{(x+1)(x+2)}\right]$

(ii) Express $\frac{x^2+7x-6}{(x-1)(2x-1)}$ in proper fraction. $\left[\frac{1}{2}+\frac{17x-13}{2(x-1)(2x-1)}\right]$

1.3 Finding partial fractions of proper fraction

You should carry out the following steps:

STEP 1:
Check that the algebraic fraction is **proper** or converted to sum of polynomial and a **proper** fraction.

STEP 2:
Factorise the denominator **completely**.
When you have factorised the denominator the factors can take various forms. You must study these forms carefully.
For example you may find $(2x-3)(2x+1)$.
These factors are both referred to as linear factors. A linear factor has the form ($ax + b$) where $a, b \in \mathbb{R}$

The linear factors could be the same, as in ($2x$ + 1)($2x$ + 1), that is $(2x+1)^2$.

This is called a repeated linear factor. Such a factor has the form $(ax+b)^2$.

Another possible form is $\left(x^2+2x+3\right)$. This is a quadratic factor which cannot be factorised into linear factors. Such a factor has the form $\left(ax^2+bx+c\right)$.

How do you know that the quadratic factor cannot be factorised further into linear factors?

It is essential that you examine the factors carefully to see which type you have. **The form that the partial fractions take depends on the type of factors obtained.**
You should examine the factors of the denominator to decide which sorts of partial fraction you will need.

<u>STEP 3</u> (See section 1.4 - rules of partial fractions):
Find the unknown constants, *A*, *B*, . . . of the partial fraction. This is done by
(a) substituting specific values for *x* or
(b) equating coefficients of like terms.

1.4 Rules of partial fractions

The following rules will apply depending on the <u>type of factors in the denominator.</u>

RULE 1: DISTINCT LINEAR FACTORS

To every linear factor ($ax + b$) in the denominator of a proper fraction there exists a partial fraction of the form $\frac{A}{(ax+b)}$, where A $\in \mathbb{R}$

Example 1.2: Express $\frac{3x}{(x+1)(x-2)}$ in partial fractions.

Solution:

Let $$\frac{3x}{(x+1)(x-2)} = \frac{A}{(x+1)} + \frac{B}{(x-2)}$$

$$\Rightarrow \quad \frac{3x}{(x+1)(x-2)} = \frac{A(x-2)+B(x+1)}{(x+1)(x-2)}$$

$$\Rightarrow \quad 3x = A(x-2)+B(x+1)$$

When
$x=-1$: $-3 = -3A \Rightarrow A = 1$
$x=2$: $6 = 3B \Rightarrow B = 2$

values of x substituted is based on the root of each linear factor

$$\therefore \frac{3x}{(x+1)(x-2)} = \frac{1}{(x+1)} + \frac{2}{(x-2)}$$

Example 1.3: Express $\frac{2x^2+3x+4}{(1-2x)(1-x^2)}$ in partial fractions.

Solution:

Let $$\frac{2x^2+3x+4}{(1-2x)(1-x^2)} = \frac{2x^2+3x+4}{(1-2x)(1-x)(1+x)} = \frac{A}{1-2x} + \frac{B}{1-x} + \frac{C}{1+x}$$

Denominator must be completely factorised

$$\Rightarrow \quad 2x^2+3x+4 = A(1-x)(1+x)+B(1-2x)(1+x)+C(1-2x)(1-x)$$

When

$x=1$: $9 = -2B$ $\Rightarrow$ $B = -\dfrac{9}{2}$

$x=-1$: $3 = 6C$ $\Rightarrow$ $C = \dfrac{1}{2}$

$x=\dfrac{1}{2}$: $6 = \dfrac{3}{4}A$ $\Rightarrow$ $A = 8$

$$\therefore \quad \frac{2x^2+3x+4}{(1-2x)(1-x^2)} = \frac{8}{1-2x} - \frac{9}{2(1-x)} + \frac{1}{2(1+x)}$$

Example 1.4: Express $\dfrac{x^3-1}{(x+1)(x+2)}$ in partial fractions.

Solution:

Using long division,

$$\frac{x^3-1}{(x+1)(x+2)} = (x-3) + \frac{7x+5}{(x+1)(x+2)}$$

Let $\dfrac{7x+5}{(x+1)(x+2)} = \dfrac{A}{x+1} + \dfrac{B}{x+2}$

$\Rightarrow \quad 7x+5 = A(x+2)+B(x+1)$

When

$x=-1$: $-2=A$ $\Rightarrow$ $A=-2$

$x=-2$: $-9=-B$ $\Rightarrow$ $B=9$

$$\therefore \quad \frac{x^3-1}{(x+1)(x+2)} = (x-3) - \frac{2}{x+1} + \frac{9}{x+2}$$

Exercise 1.5:

Express in partial fractions

(i) $\dfrac{13x+2}{x^2+x-2}$ $\left[\dfrac{8}{x+2} + \dfrac{5}{x-1}\right]$

(ii) $\dfrac{3x^2+8x-4}{(4+x)(4-x^2)}$ $\left[-\dfrac{1}{x+4} - \dfrac{1}{x+2} + \dfrac{1}{2-x}\right]$

(iii) $\dfrac{x^2+7x-6}{(x-1)(2x-1)}$ $\left[\dfrac{1}{2} + \dfrac{2}{x-1} + \dfrac{9}{2(2x-1)}\right]$

RULE 2: REPEATED LINEAR FACTORS

To every linear factor $(ax+b)$ occurring *n* times in the denominator there exists a sum of *n* partial fractions $\frac{A_1}{(ax+b)}+\frac{A_2}{(ax+b)^2}+\frac{A_3}{(ax+b)^3}+............+\frac{A_n}{(ax+b)^n}$, where $A_1, A_2, A_3, ..., A_n \in \mathbb{R}$.

Example 1.6: Express $\frac{2x^2-7}{(x+1)(x-3)^2}$ in partial fractions.

Solution:

Let $\frac{2x^2-7}{(x+1)(x-3)^2}=\frac{A}{x+1}+\frac{B}{x-3}+\frac{C}{(x-3)^2}$

$\Rightarrow \quad 2x^2-7 = A(x-3)^2+B(x-3)(x+1)+C(x+1)$

When

$x=-1: \quad -5=16A \quad \Rightarrow \quad A = -\frac{5}{16}$

$x=3: \quad 11 = 4C \quad \Rightarrow \quad C = \frac{11}{4}$

Equate coefficient of x^2 :

$2 = A + B \quad \Rightarrow \quad 2 = -\frac{5}{16} + B \quad \Rightarrow \quad B = \frac{37}{16}$

$$\therefore \quad \frac{2x^2-7}{(x+1)(x-3)^2}=-\frac{5}{16(x+1)}+\frac{37}{16(x-3)}+\frac{11}{4(x-3)^2}$$

Exercise 1.7:
Express in partial fractions

(i) $\frac{x^2-2}{(x-4)(x^2-6x+8)}$ $\left[\frac{1}{2(x-2)}+\frac{1}{2(x-4)}+\frac{7}{(x-4)^2}\right]$

RULE 3: DISTINCT QUADRATIC FACTOR

To every quadratic factor $\left(ax^2+bx+c\right)$ in the denominator there exists a partial fraction of the form $\dfrac{Ax+B}{ax^2+bx+c}$.

Example 1.8: Express $\dfrac{2x^2+1}{(x+1)\left(x^2+x+2\right)}$ in partial fractions.

Solution:

Let $\dfrac{2x^2+1}{(x+1)\left(x^2+x+2\right)}=\dfrac{A}{x+1}+\dfrac{Bx+C}{x^2+x+2}$

$\Rightarrow \quad 2x^2+1 = A(x^2+x+2)+(Bx+C)(x+1)$

When

$x=-1$: $\quad 3 = 2A \quad \Rightarrow \quad A = \dfrac{3}{2}$

Equate coefficient of x^2 :

$2 = A + B \quad \Rightarrow \quad 2 = \dfrac{3}{2} + B \quad \Rightarrow \quad B = \dfrac{1}{2}$

Equate constant term:

$1 = 2A + C \quad \Rightarrow \quad 1 = 2\left(\dfrac{3}{2}\right)+C \quad \Rightarrow \quad C=-2$

$$\therefore \quad \frac{2x^2+1}{(x+1)\left(x^2+x+2\right)}=\frac{3}{2(x+1)}+\frac{(1/2)x-2}{(x^2+x+2)}=\frac{3}{2(x+1)}+\frac{x-4}{2(x^2+x+2)}$$

Example 1.9: Express $\dfrac{2x}{x^3+1}$ in partial fractions.

Solution:

Let $f(x)=x^3+1$

Since $f(-1)=-1+1=0$ therefore $(x+1)$ is factor

By long division $f(x)=x^3+1=(x+1)\left(x^2-x+1\right)$

Let $\dfrac{2x}{x^3+1}=\dfrac{2x}{(x+1)(x^2-x+1)}=\dfrac{A}{x+1}+\dfrac{Bx+C}{(x^2-x+1)}$

$\Rightarrow \quad 2x=A(x^2-x+1)+(Bx+C)(x+1)$

When

$x=-1$: $\quad -2 = 3A \quad \Rightarrow \quad A = -\dfrac{2}{3}$

Equate coefficient of x^2 :

$$0 = A + B \quad \Rightarrow \quad 0 = -\frac{2}{3} + B \quad \Rightarrow \quad B = \frac{2}{3}$$

Equate constant term:

$$0 = A + C \quad \Rightarrow \quad 0 = -\frac{2}{3} + C \quad \Rightarrow \quad C = \frac{2}{3}$$

$$\therefore \quad \frac{2x}{x^3+1} = -\frac{2}{3(x+1)} + \frac{2x+2}{3(x^2-x+1)}$$

Exercise 1.10:

Express in partial fractions $\dfrac{x^2-x+3}{(x-2)(x^2+1)}$ $\left[\dfrac{1}{x-2} - \dfrac{1}{x^2+1}\right]$

NOTE: The following Partial Fractions decompositions are given in your Mathematical Formulae booklet:

Non-repeated linear factors:

$$\frac{px+q}{(ax+b)(cx+d)} = \frac{A}{(ax+b)} + \frac{B}{(cx+d)}$$

Repeated linear factors:

$$\frac{px^2+qx+r}{(ax+b)(cx+d)^2} = \frac{A}{(ax+b)} + \frac{B}{(cx+d)} + \frac{C}{(cx+d)^2}$$

Non-repeated quadratic factor:

$$\frac{px^2+qx+r}{(ax+b)(x^2+c^2)} = \frac{A}{(ax+b)} + \frac{Bx+C}{(x^2+c^2)}$$