

Chapter 2

Binomial Theorem

In this unit, students will:

- (a) use the binomial expansion of $(1+x)^n$ for any rational n , and the condition that $|x| < 1$ for the convergence of the binomial series.
- (b) find the coefficient of x^n from the binomial series expansion.
- (c) use the binomial expansion of $(a+bx)^n$ for any rational n , and the condition that $|x| < \left|\frac{a}{b}\right|$ for the convergence of the binomial series for small x .
- (d) use the binomial expansion of $(a+bx)^n$ for any rational n , and the condition that $|x| > \left|\frac{a}{b}\right|$ for the convergence of the binomial series for large x .

2.1 Introduction

Any expression such as $(a+b)$ involving two terms is known as a **binomial expression**. Thus $(a+b)^n$ is a binomial function, and the statement of its expansion in power of a (or b) is known as the **binomial theorem**.

Consider the expansion of $(a+b)^n$, where n is a positive integer.

$$(a+b)^0 = 1$$

$$(a+b)^1 = a+b$$

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

Notice that the coefficients of terms in the above expansions form the Pascal's triangle.

$n = 0$						1										
$n = 1$						1		1								
$n = 2$						1		2		1						
$n = 3$						1		3		3		1				
$n = 4$						1		4		6		4		1		
$n = 5$						1		5		10		10		5		1

Each row of Pascal's triangle starts and ends with a 1; others can be obtained by adding the two terms on either side of it in the preceding row.

□

2.2 The Binomial Expansion for $n \in \mathbb{Z}^+$

In general, if n is a positive integer, the expansion of $(a + b)^n$ is given by

$$\begin{aligned} (a + b)^n &= a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \binom{n}{3} a^{n-3} b^3 + \dots + \binom{n}{r} a^{n-r} b^r + \dots + \binom{n}{n-1} a b^{n-1} + b^n \\ &= a^n + n a^{n-1} b + \frac{n(n-1)}{2!} a^{n-2} b^2 + \frac{n(n-1)(n-2)}{3!} a^{n-3} b^3 + \dots + \frac{n(n-1)(n-2)\dots(n-r+1)}{r!} a^{n-r} b^r \\ &\quad + \dots + n a b^{n-1} + b^n \end{aligned}$$

This is called the Binomial Theorem for any positive integral value of n .

Notes:

(i) The Factorial Notation

Definition

The product of all the integers from 1 to n inclusive is called n factorial, i.e. $n \times (n-1) \times (n-2) \times \dots \times 2 \times 1 = n!$

For example $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$.

In particular, we define $0! = 1$.

(ii) The binomial coefficient ${}^n C_r$, is given by

$${}^n C_r = \binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)(n-2)\dots(n-r+1)}{r!}$$

(iii) The expansion of $(a + b)^n$ consists of $(n + 1)$ terms. The sum of the powers of a and b in each term is n .

$$\text{(iv)} \quad \binom{n}{r} = \binom{n}{n-r}$$

$$\therefore \binom{n}{0} = \binom{n}{n} = 1; \quad \binom{n}{1} = \binom{n}{n-1}; \quad \binom{4}{1} = \binom{4}{3} = 4$$

(v) The $(r + 1)^{\text{th}}$ term, $T_{r+1} = \binom{n}{r} a^{n-r} b^r$. i.e. the coefficient of $b^r = \binom{n}{r} a^{n-r}$

(vi) To expand $(a - b)^n$, we rewrite it as $(a + (-b))^n$

$$(a - b)^n$$

$$(a + (-b))^n$$

$$\begin{aligned}
&= a^n + \binom{n}{1} a^{n-1}(-b) + \binom{n}{2} a^{n-2}(-b)^2 + \binom{n}{3} a^{n-3}(-b)^3 + \dots + \binom{n}{r} a^{n-r}(-b)^r + \dots + \binom{n}{n-1} a(-b)^{n-1} + (-b)^n \\
&= a^n - \binom{n}{1} a^{n-1}b + \binom{n}{2} a^{n-2}b^2 - \binom{n}{3} a^{n-3}b^3 + \dots + \binom{n}{r} a^{n-r}(-b)^r + \dots + \binom{n}{n-1} a(-b)^{n-1} + (-b)^n
\end{aligned}$$

Notice that the terms are alternatively positive and negative.

Example 2.1

Expand $(2x + 3y)^5$ and simplify each term.

Solution

$$\begin{aligned}
(2x + 3y)^5 &= (2x)^5 + 5(2x)^4(3y) + 10(2x)^3(3y)^2 + 10(2x)^2(3y)^3 + 5(2x)(3y)^4 + (3y)^5 \\
&= 32x^5 + 240x^4y + 720x^3y^2 + 1080x^2y^3 + 810xy^4 + 243y^5
\end{aligned}$$

Example 2.2

Expand $\left(2x - \frac{1}{x}\right)^4$ and simplify each term.

Solution

$$\begin{aligned}
&\left(2x - \frac{1}{x}\right)^4 \\
&\left(2x + \left(-\frac{1}{x}\right)\right)^4 \\
&= (2x)^4 + \binom{4}{1}(2x)^3\left(-\frac{1}{x}\right) + \binom{4}{2}(2x)^2\left(-\frac{1}{x}\right)^2 + \binom{4}{3}(2x)\left(-\frac{1}{x}\right)^3 + \left(-\frac{1}{x}\right)^4 \\
&= (2x)^4 - \binom{4}{1}(2x)^3\left(\frac{1}{x}\right) + \binom{4}{2}(2x)^2\left(\frac{1}{x}\right)^2 - \binom{4}{3}(2x)\left(\frac{1}{x}\right)^3 + \left(\frac{1}{x}\right)^4 \\
&= 16x^4 - 32x^2 + 24 - \frac{8}{x^2} + \frac{1}{x^4}
\end{aligned}$$

Example 2.3

Expand $(2 + x)^5$ in ascending powers of x .

Solution

$$\begin{aligned}
(2 + x)^5 &= 2^5 + 5(2)^4x + 10(2)^3x^2 + 10(2)^2x^3 + 5(2)x^4 + x^5 \\
&= 32 + 80x + 80x^2 + 40x^3 + 10x^4 + x^5
\end{aligned}$$

Or

$$\begin{aligned}
(2 + x)^5 &= 2^5 \left(1 + \frac{x}{2}\right)^5 = 32 \left[1 + 5\left(\frac{x}{2}\right) + 10\left(\frac{x}{2}\right)^2 + 10\left(\frac{x}{2}\right)^3 + 5\left(\frac{x}{2}\right)^4 + \left(\frac{x}{2}\right)^5\right] \\
&= 32 + 80x + 80x^2 + 40x^3 + 10x^4 + x^5
\end{aligned}$$

Or

$$\begin{aligned}(2+x)^5 &= x^5 \left(1 + \frac{2}{x}\right)^5 = x^5 \left[1 + 5\left(\frac{2}{x}\right) + 10\left(\frac{2}{x}\right)^2 + 10\left(\frac{2}{x}\right)^3 + 5\left(\frac{2}{x}\right)^4 + \left(\frac{2}{x}\right)^5\right] \\ &= x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32\end{aligned}$$

Example 2.4

Find the first three terms in the expansion, in ascending powers of x , of $(1+2x+3x^2)^8$.

Solution

$$(1+2x+3x^2)^8 = [1 + (2x + 3x^2)]^8 = 1 + \binom{8}{1}(2x + 3x^2) + \binom{8}{2}(2x + 3x^2)^2 + \dots$$

$$= 1 + 8(2x + 3x^2) + 28(2x + 3x^2)^2 + \dots$$

$$= 1 + (16x + 24x^2) + (112x^2 + \dots)$$

$$= 1 + 16x + 136x^2 + \dots$$

Example 2.5

Find the term involving x^3 in the expansion of $\left(x^3 + \frac{1}{x}\right)^{13}$.

Solution

Recall that $T_{r+1} = \binom{n}{r} a^{n-r} b^r$.

The T_{r+1} of above expansion is $\binom{13}{r} (x^3)^{13-r} \left(\frac{1}{x}\right)^r = \binom{13}{r} x^{39-3r} x^{-r} = \binom{13}{r} x^{39-4r}$

$$\text{Set } 39 - 4r = 3 \quad \therefore r = 9$$

Thus the term required is $\binom{13}{9} x^3 = 715x^3$

Example 2.6

Calculate

(i) the term independent of x and

(ii) the coefficient of x^8 in the expansion of $\left(2x^3 - \frac{1}{x}\right)^{12}$.

Solution

(i) The $(r+1)^{\text{th}}$ term is

$$\begin{aligned}T_{r+1} &= \binom{12}{r} (2x^3)^{12-r} \left(-\frac{1}{x}\right)^r = \binom{12}{r} (2)^{12-r} (x)^{36-3r} (-1)^r (x)^{-r} \\ &= \binom{12}{r} (-1)^r (2^{12-r}) (x^{36-4r})\end{aligned}$$

For the term independent of x , set $36 - 4r = 0 \Rightarrow r = 9$

\therefore The term independent of x is $T_{10} = \binom{12}{9} (-1)^9 (2^{12-9}) = -(220)(8) = -1760$

(ii) The coefficient of x^8 requires $36 - 4r = 8 \Rightarrow r = 7$

$$\Rightarrow T_8 = \binom{12}{7} (-1)^7 (2^{12-7}) (x^{36-28}) = -(792)(32) x^8 = -25344 x^8$$

\therefore The coefficient of x^8 is -25344 .

Exercise 2.7

Find, in the expansion of $(1 + 2x) \left(1 - \frac{1}{x}\right)^5$,

- (i) the term independent of x , [−9]
- (ii) the coefficient of x^{-1} [15]

2.3 Binomial Expansion for n where n is a negative integer or a fraction

2.3.1 Binomial Expansion of $(1 + x)^n$

The binomial expansion of $(1 + x)^n$, in ascending powers of x (ie. when x is small) is given by

$$\begin{aligned} (1 + x)^n &= 1^n + \frac{n}{1!} (1)^{n-1} x^1 + \frac{n(n-1)}{2!} (1)^{n-2} x^2 + \frac{n(n-1)(n-2)}{3!} (1)^{n-3} x^3 + \dots + \frac{n(n-1)\dots(n-r+1)}{r!} (1)^{n-r} x^r + \dots \\ &= 1 + nx^1 + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \dots + \frac{n(n-1)\dots(n-r+1)}{r!} x^r + \dots \end{aligned}$$

This is only valid for $|x| < 1$, i.e., $-1 < x < 1$.

(This formula can also be found in the formula sheet.)

This expansion contains an infinite number of terms and is only valid if it is convergent, i.e., successive terms of the series must be smaller.

Note:

(i) The symbols $\binom{n}{r}$ are not used here as they become meaningless.

(ie. $\binom{-1}{2}$ and $\binom{1/2}{3}$ are undefined.)

When n is a fraction or a negative integer, it is important that we state the range of values of x for which the expansion is valid, i.e., the range of values of x for which the expansion works. (See Example 2.17)

Example 2.8

Expand $\frac{1}{(1-x)^2}$ as a series of ascending powers of x , up to and including the terms in x^4 and state its valid range of x .

Solution:

$$\begin{aligned}\frac{1}{(1-x)^2} &= (1-x)^{-2} \\ &= 1 + (-2)(-x) + \frac{(-2)(-3)}{2!}(-x)^2 + \frac{(-2)(-3)(-4)}{3!}(-x)^3 + \frac{(-2)(-3)(-4)(-5)}{4!}(-x)^4 + \dots \\ &= 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \dots\end{aligned}$$

valid range of x is $|-x| < 1 \Leftrightarrow |x| < 1$.

Exercise 2.9

Expand the following as a series of ascending powers of x , up to and including the terms in x^4 and state its valid range of x .

$$(i) \frac{1}{(1-x)} \qquad [1 + x + x^2 + x^3 + x^4 + \dots; |x| < 1]$$

$$(ii) \frac{1}{(1+x)^2} \qquad [1 - 2x + 3x^2 - 4x^3 + 5x^4 + \dots; |x| < 1]$$

Example 2.10

Expand $(1+2x)^{\frac{3}{2}}$ as a series of ascending powers of x , up to and including the terms in x^4 and state its valid range of x .

Solution:

$$\begin{aligned}(1+2x)^{\frac{3}{2}} &= 1 + \frac{3}{2}(2x) + \frac{(\frac{3}{2})(\frac{3}{2}-1)}{2!}(2x)^2 + \frac{(\frac{3}{2})(\frac{3}{2}-1)(\frac{3}{2}-2)}{3!}(2x)^3 + \frac{(\frac{3}{2})(\frac{3}{2}-1)(\frac{3}{2}-2)(\frac{3}{2}-3)}{4!}(2x)^4 + \dots \\ &= 1 + 3x + \frac{3}{2}x^2 - \frac{1}{2}x^3 + \frac{3}{8}x^4 + \dots\end{aligned}$$

valid range of x is $|2x| < 1 \Rightarrow |x| < \frac{1}{2}$

Example 2.11

Expand $(1-5x)^{-\frac{1}{2}}$ as a series of ascending powers of x , up to and including the terms in x^4 and state its valid range of x .

Solution:

$$\begin{aligned}(1-5x)^{-\frac{1}{2}} &= 1 + (-\frac{1}{2})(-5x) + \frac{(-\frac{1}{2})(-\frac{1}{2}-1)}{2!}(-5x)^2 + \frac{(-\frac{1}{2})(-\frac{1}{2}-1)(-\frac{1}{2}-2)}{3!}(-5x)^3 \\ &\quad + \frac{(-\frac{1}{2})(-\frac{1}{2}-1)(-\frac{1}{2}-2)(-\frac{1}{2}-3)}{4!}(-5x)^4 + \dots\end{aligned}$$

$$= 1 + \frac{5}{2}x + \frac{75}{8}x^2 + \frac{625}{16}x^3 + \frac{21875}{128}x^4 + \dots$$

valid range of x is $|5x| < 1 \Rightarrow |x| < \frac{1}{5}$

Exercise 2.12

Give the expansion in ascending powers of x , up to the 4th term, of $(1 + 3x)^{\frac{1}{3}}$. State the range of values of x for which the expansion is valid.

$$\left[1 + x - x^2 + \frac{5}{3}x^3 + \dots; -\frac{1}{3} < x < \frac{1}{3}\right]$$

2.3.2 Binomial Expansion of $(a + bx)^n$ where $a \neq 1$

For the expansion to be in ascending powers of x ,

$$\begin{aligned} (a + bx)^n &= a^n \left(1 + \frac{bx}{a}\right)^n \\ &= a^n \left[1 + \frac{n}{1!} \left(\frac{bx}{a}\right) + \frac{n(n-1)}{2!} \left(\frac{bx}{a}\right)^2 + \frac{n(n-1)(n-2)}{3!} \left(\frac{bx}{a}\right)^3 + \dots \right. \\ &\quad \left. + \frac{n(n-1)(n-2)\dots(n-r+1)}{r!} \left(\frac{bx}{a}\right)^r + \dots\right] \end{aligned}$$

Note:

- (i) The expansion is in ascending powers of x .
- (ii) We factor out the constant term a before expansion. This expansion is valid or convergent

$$\text{if } \left|\frac{bx}{a}\right| < 1 \quad \Rightarrow \quad |x| < \left|\frac{a}{b}\right| \quad \Rightarrow \quad -\left|\frac{a}{b}\right| < x < \left|\frac{a}{b}\right|.$$

- (iii) To expand $(a - bx)^n$, just rewrite it as $(a + (-bx))^n$

Example 2.13

Expand $(2 + 3x)^{-2}$ as a series of ascending powers of x , up to and including the terms in x^3 and find its valid range of x .

Solution:

$$\begin{aligned} (2 + 3x)^{-2} &= 2^{-2} \left[1 + \frac{3x}{2}\right]^{-2} = \frac{1}{4} \left[1 + \frac{(-2)}{1!} \left(\frac{3x}{2}\right) + \frac{(-2)(-3)}{2!} \left(\frac{3x}{2}\right)^2 + \frac{(-2)(-3)(-4)}{3!} \left(\frac{3x}{2}\right)^3 + \dots\right] \\ &= \frac{1}{4} \left[1 - 3x + \frac{27}{4}x^2 - \frac{108}{8}x^3 + \dots\right] \end{aligned}$$

The expansion is valid if $\left|\frac{3x}{2}\right| < 1 \quad \Rightarrow \quad |x| < \frac{2}{3} \quad \Rightarrow \quad -\frac{2}{3} < x < \frac{2}{3}$

Exercise 2.14

Find the expansion in ascending powers of x , up to and including the term in x^3 , of $(2 + x)^{-2}$, simplifying the coefficients. State the range of x for which the expansion is valid.

$$\left[\frac{1}{4} - \frac{1}{4}x + \frac{3}{16}x^2 - \frac{1}{8}x^3 + \dots; -2 < x < 2\right]$$

For the expansion in descending powers of x /ascending powers of $\frac{1}{x}$.

In some cases, you are told to expand in descending powers of x

$$\begin{aligned}(a+bx)^n &= (bx)^n \left(1 + \frac{a}{bx}\right)^n \\ &= (bx)^n \left[1 + \frac{n}{1!} \left(\frac{a}{bx}\right) + \frac{n(n-1)}{2!} \left(\frac{a}{bx}\right)^2 + \frac{n(n-1)(n-2)}{3!} \left(\frac{a}{bx}\right)^3 + \dots \right. \\ &\quad \left. + \frac{n(n-1)(n-2)\dots[n-r+1]}{r!} \left(\frac{a}{bx}\right)^r + \dots\right]\end{aligned}$$

Note:

(i) We factor out the term (bx) before expansion.

(ii) The expansion is valid or convergent

$$\text{if } \left|\frac{a}{bx}\right| < 1 \quad \Rightarrow \quad |x| > \left|\frac{a}{b}\right| \quad \Rightarrow \quad x < -\left|\frac{a}{b}\right| \quad \text{or} \quad x > \left|\frac{a}{b}\right|.$$

Example 2.15

Expand $(2+3x)^{-2}$ as a series of descending powers of x , up to and including the terms in $\frac{1}{x^3}$ and find its valid range of x .

Solution:

$$\begin{aligned}(2+3x)^{-2} &= (3x)^{-2} \left[1 + \frac{2}{3x}\right]^{-2} \\ &= \frac{1}{9x^2} \left[1 + \frac{(-2)}{1!} \left(\frac{2}{3x}\right) + \frac{(-2)(-3)}{2!} \left(\frac{2}{3x}\right)^2 + \frac{(-2)(-3)(-4)}{3!} \left(\frac{2}{3x}\right)^3 + \dots\right] \\ &= \frac{1}{9x^2} \left[1 - \frac{4}{3x} + \frac{4}{3x^2} - \frac{32}{27x^3} + \dots\right] \\ &= \frac{1}{9x^2} - \frac{4}{27x^3} + \dots\end{aligned}$$

$$\text{The expansion is valid if } \left|\frac{2}{3x}\right| < 1 \quad \Rightarrow \quad |x| > \frac{2}{3} \quad \Rightarrow \quad x < -\frac{2}{3} \quad \text{or} \quad x > \frac{2}{3}$$

Exercise 2.16

Given that $f(x) = \frac{1}{x-2} - \frac{1}{x^2+1}$, find the series expansion in descending powers of x of $f(x)$ up to and including the 3rd non-zero term.

Write down the restriction that must be imposed on x for the expansion to be valid.

$$[x^{-1} + x^{-2} + 4x^{-3} + \dots; x < -2 \quad \text{or} \quad x > 2]$$

Example 2.17

Evaluate to 4 decimal places the value of $\sqrt{26}$.

[Note: Using calculator, $\sqrt{26} \approx 5.099019514\dots$]

Solution:

$$\begin{aligned} \sqrt{26} &= (5^2 + 1)^{1/2} = (5^2)^{1/2} \left(1 + \frac{1}{25}\right)^{1/2} = 5 \left[1 + \frac{1}{25}\right]^{1/2}, \quad \text{where } \left|\frac{1}{25}\right| < 1 \\ &= 5 \left[1 + \frac{1/2}{1!} \left(\frac{1}{25}\right) + \frac{(1/2)(-1/2)}{2!} \left(\frac{1}{25}\right)^2 + \frac{(1/2)(-1/2)(-3/2)}{3!} \left(\frac{1}{25}\right)^3 + \dots\right] \\ &= 5 + 0.1000 - 0.0010 + 0.00002 - \dots = 5.1034\dots \end{aligned}$$

But if we express it in this manner

$$\begin{aligned} \sqrt{26} &= (1 + 5^2)^{1/2}, \quad \text{but } |25| > 1????? \\ &= 1 + \frac{1/2}{1!} (25) + \frac{(1/2)(-1/2)}{2!} (25)^2 + \dots = 1 + 12.5 - 78.125 + 976.5626 - \dots \\ &= \text{invalid answer} \end{aligned}$$

More examples

Example 2.18

Expand $(1 + x)^{-1} + (1 - x)^{-2}$ as a series of ascending powers of x , up to and including the terms in x^3 . Hence, find the coefficient of x^n .

Solution:

$$\begin{aligned} (1 + x)^{-1} &= 1 + (-1)x + \frac{(-1)(-2)}{2!} x^2 + \frac{(-1)(-2)(-3)}{3!} x^3 + \dots \\ &= 1 - x + x^2 - x^3 + \dots + (-1)^n x^n + \dots \end{aligned}$$

[Note: Look for a pattern in the series expansion]

$$\begin{aligned} (1 - x)^{-2} &= 1 + (-2)(-x) + \frac{(-2)(-3)}{2!} (-x)^2 + \frac{(-2)(-3)(-4)}{3!} (-x)^3 + \dots \\ &= 1 + 2x + 3x^2 + 4x^3 + \dots + (n + 1)x^n + \dots \end{aligned}$$

Hence,

$$\begin{aligned} (1 + x)^{-1} + (1 - x)^{-2} &= (1 - x + x^2 - x^3 + \dots) + (1 + 2x + 3x^2 + 4x^3 + \dots) \\ &= 2 + x + 4x^2 + 3x^3 + \dots \end{aligned}$$

The coefficient of x^n is $= (-1)^n + (n + 1)$.

Example 2.19

Expand $\frac{9 + 11x + 3x^2}{(3 + x)(2 + x)^2}$ as a series of ascending powers of x , up to and including the terms in x^3 .

Hence, find the coefficient of x^n . State the valid range of x .

Solution:

$$\text{Let } f(x) = \frac{9 + 11x + 3x^2}{(3+x)(2+x)^2} = \frac{A}{3+x} + \frac{B}{(2+x)} + \frac{C}{(2+x)^2}$$

$$\Rightarrow 9 + 11x + 3x^2 = A(2+x)^2 + B(3+x)(2+x) + C(3+x)$$

$$\text{When } x = -2: \quad \Rightarrow \quad -1 = C$$

$$\text{When } x = -3: \quad \Rightarrow \quad 3 = A$$

$$\text{Equating the coefficient of } x^2: \quad 3 = A + B \quad \Rightarrow \quad 3 = 3 + B \quad \Rightarrow \quad B = 0$$

$$\begin{aligned} f(x) &= \frac{3}{3+x} - \frac{1}{(2+x)^2} \\ &= 3(3+x)^{-1} - (2+x)^{-2} \\ &= 3(3)^{-1} \left(1 + \frac{x}{3}\right)^{-1} - (2)^{-2} \left(1 + \frac{x}{2}\right)^{-2} \\ &= \left(1 + \frac{x}{3}\right)^{-1} - \frac{1}{4} \left(1 + \frac{x}{2}\right)^{-2} \\ &= \left[1 + (-1) \binom{-1}{1} \left(\frac{x}{3}\right) + \frac{(-1)(-2)}{2!} \left(\frac{x}{3}\right)^2 + \frac{(-1)(-2)(-3)}{3!} \left(\frac{x}{3}\right)^3 + \dots \right] \\ &\quad - \frac{1}{4} \left[1 + (-2) \binom{-2}{1} \left(\frac{x}{2}\right) + \frac{(-2)(-3)}{2!} \left(\frac{x}{2}\right)^2 + \frac{(-2)(-3)(-4)}{3!} \left(\frac{x}{2}\right)^3 + \dots \right] \\ &= 1 - \frac{x}{3} + \frac{x^2}{9} - \frac{x^3}{27} + \dots - \frac{1}{4} \left(1 - x + \frac{3x^2}{4} - \frac{x^3}{2} + \dots \right) \\ &= \frac{3}{4} - \frac{1}{12}x - \frac{11}{144}x^2 + \frac{19}{216}x^3 + \dots \end{aligned}$$

To find the coefficient of x^n , we write out the terms individually for each of the binomial expansions.

Consider the expansion $\left(1 + \frac{x}{3}\right)^{-1}$. Then,

$$1 + (-1) \binom{-1}{1} \left(\frac{x}{3}\right) + \frac{(-1)(-2)}{1 \times 2} \binom{-1}{2} \left(\frac{x}{3}\right)^2 + \frac{(-1)(-2)(-3)}{1 \times 2 \times 3} \binom{-1}{3} \left(\frac{x}{3}\right)^3 + \frac{(-1)(-2)(-3)(-4)}{1 \times 2 \times 3 \times 4} \binom{-1}{4} \left(\frac{x}{3}\right)^4 + \frac{(-1)(-2)(-3)(-4)(-5)}{1 \times 2 \times 3 \times 4 \times 5} \binom{-1}{5} \left(\frac{x}{3}\right)^5 + \dots$$

Hence, the term containing x^n is $\left(1 + \frac{x}{3}\right)^{-1}$ is $(-1)^n \binom{-1}{n} \left(\frac{1}{3}\right)^n x^n$.

Consider the expansion $\left(1 + \frac{x}{2}\right)^{-2}$. Then,

$$1 + (-2) \binom{-2}{1} \left(\frac{x}{2}\right) + \frac{(-2)(-3)}{1 \times 2} \binom{-2}{2} \left(\frac{x}{2}\right)^2 + \frac{(-2)(-3)(-4)}{1 \times 2 \times 3} \binom{-2}{3} \left(\frac{x}{2}\right)^3 + \frac{(-2)(-3)(-4)(-5)}{1 \times 2 \times 3 \times 4} \binom{-2}{4} \left(\frac{x}{2}\right)^4 + \frac{(-2)(-3)(-4)(-5)(-6)}{1 \times 2 \times 3 \times 4 \times 5} \binom{-2}{5} \left(\frac{x}{2}\right)^5 + \dots$$

The term containing x^n is $\left(1 + \frac{x}{2}\right)^{-2}$ is $(-1)^n \binom{-2}{n} \left(\frac{1}{2}\right)^n x^n$.

Hence, the coefficient of x^n for $f(x)$ is $(-1)^n \left(\frac{1}{3}\right)^n - \frac{1}{4}(-1)^n (n+1) \left(\frac{1}{2}\right)^n$.

To find the valid range of x

$$\text{ie. } \left|\frac{x}{3}\right| < 1 \quad \text{and} \quad \left|\frac{x}{2}\right| < 1 \quad \Rightarrow \quad |x| < 3 \quad \text{and} \quad |x| < 2 \quad \Rightarrow \quad |x| < 2$$

\therefore The valid range of x is $|x| < 2$ i.e. $-2 < x < 2$

Alternative method of finding the x^n term:

The term containing x^n is $\left(1 + \frac{x}{3}\right)^{-1}$ is $(-1)^n \left(\frac{1}{3}\right)^n x^n$.

(from observing that the expansion that each term increases by a factor of $\left(-\frac{1}{3}\right)$.)

The term containing x^n is $\left(1 + \frac{x}{2}\right)^{-2}$ is

$$\begin{aligned} \frac{(-2)(-3)(-4)\dots(-n)(-n-1)}{n!} \left(\frac{x}{2}\right)^n &= \frac{(2)(3)(4)\dots(n)(n+1)}{n!} (-1)^n \left(\frac{1}{2}\right)^n (x^n) \\ &= (-1)^n (n+1) \left(\frac{1}{2}\right)^n x^n \end{aligned}$$

Hence, the coefficient of x^n for $f(x)$ is $(-1)^n \left(\frac{1}{3}\right)^n - \frac{1}{4}(-1)^n (n+1) \left(\frac{1}{2}\right)^n$.

Example 2.20

(Cont'd from Exercise 2.14)

Find the expansion in ascending powers of x , up to and including the term in x^3 , of $(2+x)^{-2}$, simplifying the coefficients. State the range of x for which the expansion is valid.

Find, in terms of n , the coefficient of x^n in the expansion.

$$\left[\frac{1}{4} - \frac{1}{4}x + \frac{3}{16}x^2 - \frac{1}{8}x^3 + \dots; -2 < x < 2; (-1)^n (n+1) \left(\frac{1}{2}\right)^{n+2} \right]$$

2.4 Application of the Binomial Expansion to Approximation

Example 2.21

Evaluate $\frac{1}{\sqrt{4.08}}$ to 4 decimal places.

Solution:

$$\begin{aligned}\frac{1}{\sqrt{4.08}} &= (4.08)^{-1/2} = (4 + 0.08)^{-1/2} = 4^{-1/2} \left(1 + \frac{0.08}{4}\right)^{-1/2} = \frac{1}{2} (1 + 0.02)^{-1/2} \\ &\approx \frac{1}{2} \left[1 + \left(-\frac{1}{2}\right)(0.02) + \frac{(-1/2)(-3/2)}{2!} (0.02)^2 + \dots \text{negligible terms} \right] \\ &= \frac{1}{2} (1 - 0.01 + 0.00015) = 0.4951 \quad (4 \text{ decimal places})\end{aligned}$$

Example 2.22

Expand $\left(x + \frac{1}{4}\right)^{-\frac{1}{2}}$ in ascending powers of x , and find the first three terms in the expansion, stating the values of x for which the expansion is valid.

By putting $x = \frac{1}{8}$, find an approximate value of $\sqrt{\frac{2}{3}}$ as a fraction in its lowest term.

Solution:

$$\begin{aligned}\left(x + \frac{1}{4}\right)^{-\frac{1}{2}} &= \left(\frac{1}{4}\right)^{-\frac{1}{2}} (1 + 4x)^{-\frac{1}{2}} = 2 \left[1 + \left(-\frac{1}{2}\right)(4x) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!} (4x)^2 + \dots \right] \\ &\approx 2 - 4x + 12x^2\end{aligned}$$

Note that the expansion is valid for $|4x| < 1 \Rightarrow |x| < \frac{1}{4} \Rightarrow -\frac{1}{4} < x < \frac{1}{4}$

Putting $x = \frac{1}{8}$, we get $\left(\frac{1}{8} + \frac{1}{4}\right)^{-\frac{1}{2}} = \left(\frac{8}{3}\right)^{\frac{1}{2}} = 2\sqrt{\frac{2}{3}}$

Thus, $2\sqrt{\frac{2}{3}} \approx 2 - 4\left(\frac{1}{8}\right) + 12\left(\frac{1}{8}\right)^2 \Rightarrow \sqrt{\frac{2}{3}} \approx \frac{27}{32}$

Exercise 2.23

Show that, if terms in x^3 and higher powers of x in the expansion in ascending powers of x are

neglected, then $\left(\frac{1-x}{1+x}\right)^n \approx 1 - 2nx + 2n^2x^2$. Hence, find an approximation to the cube root of $\frac{15}{17}$,

leaving your answer in terms of a fraction.

$$\left[\frac{1105}{1152}\right]$$

Example 2.24

Expand $\frac{1}{\sqrt{2+x}}$ as a series of ascending powers of x , where $|x| < 2$, up to and including the terms in x^3 .

By substituting $x = \frac{1}{2}$ in the above expansion, show that $\sqrt{5} \approx \frac{2048}{915}$.

Solution:

$$(2+x)^{-\frac{1}{2}}$$

$$\begin{aligned}
&= 2^{-\frac{1}{2}} \left(1 + \frac{x}{2}\right)^{-\frac{1}{2}} \\
&= \frac{1}{\sqrt{2}} \left[1 + \binom{-\frac{1}{2}}{1} \left(\frac{x}{2}\right) + \frac{\binom{-\frac{1}{2}}{2} \left(\frac{x}{2}\right)^2}{2!} + \frac{\binom{-\frac{1}{2}}{3} \left(\frac{x}{2}\right)^3}{3!} + \dots \right] \\
&= \frac{1}{\sqrt{2}} \left[1 - \frac{x}{4} + \frac{3}{32}x^2 - \frac{5}{128}x^3 + \dots \right], \quad \left| \frac{x}{2} \right| < 1 \Rightarrow |x| < 2
\end{aligned}$$

$$\text{Let } x = \frac{1}{2}$$

$$\left(2 + \frac{1}{2}\right)^{-\frac{1}{2}} = \left(\frac{5}{2}\right)^{-\frac{1}{2}} = \left(\frac{2}{5}\right)^{\frac{1}{2}} = \frac{\sqrt{2}}{\sqrt{5}}$$

$$\therefore \frac{\sqrt{2}}{\sqrt{5}} \approx \frac{1}{\sqrt{2}} \left[1 - \frac{1}{4} \left(\frac{1}{2}\right) + \frac{3}{32} \left(\frac{1}{2}\right)^2 - \frac{5}{128} \left(\frac{1}{2}\right)^3 \right]$$

$$\frac{\sqrt{2}}{\sqrt{5}} \approx \frac{1}{\sqrt{2}} \left(\frac{915}{1024} \right) \Rightarrow \sqrt{5} \approx \frac{2048}{915}$$

Example 2.25

Show that, if x is so small that its cube and higher powers can be neglected, then

$\left(\frac{1+x}{1-x}\right)^{1/2} \approx 1 + x + \frac{1}{2}x^2$. By choosing a suitable value of x , find an approximation for $\sqrt{5}$, leaving your answer as a fraction.

Solution:

$$\begin{aligned}
(1+x)^{\frac{1}{2}} &= 1 + \frac{1}{2}x + \frac{\binom{\frac{1}{2}}{2} \left(\frac{1}{2}\right)^2 x^2}{2!} + \dots \\
&= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \dots
\end{aligned}$$

$$\begin{aligned}
(1-x)^{-\frac{1}{2}} &= 1 + \binom{-\frac{1}{2}}{1}(-x) + \frac{\binom{-\frac{1}{2}}{2} \left(-\frac{3}{2}\right) \left(-x\right)^2}{2!} + \dots \\
&= 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \dots
\end{aligned}$$

$$\begin{aligned}
(1+x)^{\frac{1}{2}}(1-x)^{-\frac{1}{2}} &\approx \left(1 + \frac{1}{2}x - \frac{1}{8}x^2\right) \left(1 + \frac{1}{2}x + \frac{3}{8}x^2\right) \\
&\approx 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{1}{2}x + \frac{1}{4}x^2 - \frac{1}{8}x^2 = 1 + x + \frac{1}{2}x^2
\end{aligned}$$

$$\text{Let } x = \frac{1}{9}$$

$$\left(\frac{1+\frac{1}{9}}{1-\frac{1}{9}}\right)^{\frac{1}{2}} = \left(\frac{10}{8}\right)^{\frac{1}{2}} = \left(\frac{5}{4}\right)^{\frac{1}{2}} = \frac{\sqrt{5}}{2}$$

$$\begin{aligned}\therefore \frac{\sqrt{5}}{2} &\approx 1 + \frac{1}{9} + \frac{1}{2} \left(\frac{1}{9} \right)^2 = \frac{181}{162} \\ \Rightarrow \sqrt{5} &\approx 2 \times \frac{181}{162} = \frac{181}{81}\end{aligned}$$

Are there any other possible values of x that we can use to evaluate $\sqrt{5}$? If so, which one is more accurate?

Suggested Answer:

$$\text{Let } x = \frac{2}{3}$$

$$\left(\frac{1 + \frac{2}{3}}{1 - \frac{2}{3}} \right)^{\frac{1}{2}} = \left(\frac{5}{1} \right)^{\frac{1}{2}} = \sqrt{5}$$

$$\therefore \sqrt{5} \approx 1 + \frac{2}{3} + \frac{1}{2} \left(\frac{2}{3} \right)^2 = \frac{17}{9}$$

Exercise 2.26

(i) (Cont'd from **Exercise 2.12**) Give the expansion in ascending powers of x , up to the 4th term, of $(1+3x)^{\frac{1}{3}}$. State the range of values of x for which the expansion is valid. Hence find an approximation for $\sqrt[3]{0.97}$, correct to 3 decimal places.

$$\left[1 + x - x^2 + \frac{5}{3}x^3 + \dots; -\frac{1}{3} < x < \frac{1}{3}; 0.990 \right]$$